Notes on Fourier Transforms

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We provide here an intuitive but non-rigorous treatment of Fourier transforms. The material comes from [4, 3, 2, 5]. See also Alan Chang's notes.

Audio Signals as a Function of Time As beautifully explained in [1, §50], any periodic audio signal, i.e. one that repeat itself at regular intervals, can be synthesised using a superposition of sinusoid waves of different amplitudes a_n and frequencies v_n and this can be represented using the formula

$$f(t) = \sum_{n=0}^{N} a_n \cos(2\pi v_n t) + b_n \sin(2\pi v_n t). \tag{1}$$

A non-periodic signal can be thought of as the limiting case of a periodic one, where the period tends to infinity, which means we need to approximate it with sinusoid waves with frequency tending to zero, giving us the following generalisation of (1) assuming the function $f()$ satisfy a mild smoothness condition:

$$f(t) = \int_{-\infty}^{\infty} a(v)\cos(2\pi vt)dv + \int_{-\infty}^{\infty} b(v)\sin(2\pi vt)dv \tag{2}$$

Using Euler's formula $e^{ix} = \cos x + i \sin x$, we can rewrite (2) as follows to simplify algebraic manipulations via its connection to angular geometry, where $\hat{f}(v)$ is complex-valued:

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(v) e^{2\pi ivt} dv. \tag{3}$$

Audio Signals as a Function of Frequency To arrive at the Fourier transform formula, we can pick an arbitrary frequency ξ and multiply both sides of (3) by $e^{-2\pi i\xi t}$ and then integrating over t to yield

$$\begin{aligned}
\int_{-\infty}^{\infty} f(t)e^{-2\pi i\xi t}dt &= \int_{-\infty}^{\infty} e^{-2\pi i\xi t}\left[\int_{-\infty}^{\infty} \hat{f}(v)e^{2\pi ivt}dv\right]dt &(4)\\
&= \int_{-\infty}^{\infty} \hat{f}(v)\left[\int_{-\infty}^{\infty} e^{-2\pi i\xi t}e^{2\pi ivt}dt\right]dv &(5)\\
&= \int_{-\infty}^{\infty} \hat{f}(v)\delta(v-\xi)dv &(6)\\
&= \hat{f}(\xi), &(7)
\end{aligned}$$

where δ is the Dirac delta function. Note the symmetry between these so-called Fourier transform pairs:

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(v) e^{2\pi i v t} dv \qquad \hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \xi t} dt \tag{8}$$

The key result we needed in the derivation above is

$$\int_{-\infty}^{\infty} e^{2\pi i v t} e^{-2\pi i \xi t} dt = \delta(v - \xi), \tag{9}$$

which can be thought of as a generalisation of the following for $m, n \in \mathbb{Z}$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imx} e^{-inx} dx = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases} \tag{10}$$

This can be verified using symbolic integration tools like Wolfram Alpha.

A visual intuitive explanation of Fourier transform by Elan Ness-Cohn can be found at this URL. In the following, we will give the algebraic intuition behind Fourier transforms, starting with a review of the roots of unity.

Roots of Unity

Definition 1. For any positive integer n, the n-th roots of unity are the (complex) solutions to the equation $x^n = 1$.

Example 1. The 1st root of unity is 1. The 2nd roots of unity are $\omega_2^0 = 1$ and $\omega_2^1 = -1$. The 3rd roots of unity are $\omega_3^0 = 1$, $\omega_3^1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\omega_3^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Proposition 1. The set of n-th roots of unity is given by

$$\{\omega_n^k \mid k = 0, 1, \ldots, n-1\}, \tag{11}$$

where $\omega_n = e^{2\pi i/n}$ for a positive integer k.

Proof. From Euler's formula $e^{i2\pi} = \cos 2\pi + i \sin 2\pi = 1$ and the periodicity of cos and sin, we have

$$x^n = 1 = e^0 = e^{2\pi i} = e^{4\pi i} = e^{6\pi i} = \cdots = e^{2k\pi i}.$$

Raising each term to the power of $1/n$ yields the n distinct solutions to x. □

Geometrically, we can interpret the n-th roots of unity as the points that are evenly spread on the unit circle in the complex plane, starting from 1 on the real axis which is a root of itself. Equivalently, they are the vertices of a regular n-gon that lies on the unit circle, with the real value 1 as one of the n vertices. Figure 1 illustrates the 3rd roots of unity.

For those familiar with group theory, in $\mathbb{C}$, the n-th roots of unity form a cyclic group under multiplication, with the generator $e^{2\pi i/n}$ and group order n.

Here is a key result we will need for Fourier transforms; there are different proofs for this result at `https://www.kylem.net/math/roots_unity.html`.

Proposition 2. Given the roots of unity given in (11), we have $\sum_{k=0}^{n-1} \omega_n^k = 0$.

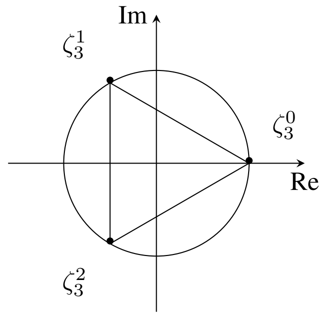

Figure 1: The 3rd roots of unity $\zeta_3^0 = 1$, $\zeta_3^1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\zeta_3^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. Note also that $\zeta_3^{-1} = \zeta_3^2$, $\zeta_3^{-2} = \zeta_3^1$, and $\zeta_3^{-3} = \zeta_3^3 = \zeta_3^0$.

Harmonics and Function Decomposition

We know from physics [1, §47-50] that audio signals are made up of a superposition of harmonics. Let's now look at a simple mathematical abstraction of harmonics and how they can be used to compose more complex functions.

Definition 2. Given an integer n, $\omega_n := e^{2\pi i/n}$ and a $j \in \{0, \dots, n-1\}$, we say a function $f : \mathbb{C} \to \mathbb{C}$ is a harmonic of order j if, for all $z \in \mathbb{C}$,

$$f(\omega_n z) = \omega_n^j f(z).$$

Example 2. Consider the case of $n = 2$. If f is a harmonic of order 0, then it satisfies $f(\omega_2 z) = \omega_2^0 f(z)$, or equivalently $f(-z) = f(z)$, which means f is an even function. Similarly, we can see that if a function g is a harmonic of order 1, then g is an odd function since $g(-z) = -g(z)$.

Now, we know any function $f(x)$ can be decomposed into the sum of an even function and an odd function as follows:

$$\begin{aligned} f(x) &= f_e(x) + f_o(x) \\ &= \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)). \end{aligned} \tag{12}$$

We can generalise that basic pattern to arrive at the following.

Proposition 3. Given an integer n and $\omega_n := e^{2\pi i/n}$, every function $f : \mathbb{C} \to \mathbb{C}$ can be uniquely written as $f(z) = \sum_{j=0}^{n-1} f_j(z)$, where each f_j defined by

$$f_j(z) = \frac{1}{n}\sum_{k=0}^{n-1} f(\omega_n^k z)\omega_n^{-jk} \tag{13}$$

is a harmonic of order j; i.e. $f_j(\omega_n z) = \omega_n^j f_j(z)$.

Proof. First, it is worth noting that, for $n = 2$, we recover (12) as a special case. Consider now the case of $n = 3$. Writing ω as a shorthand for ω_3, we have

$$\begin{aligned} f(z) = &\quad f_0(z) + f_1(z) + f_2(z) && (14)\\ = &\quad \frac{1}{3}\left[f(z) + f(\omega z)\omega^0 + f(\omega^2 z)\omega^0\right] && (15)\\ &+\frac{1}{3}\left[f(z) + f(\omega z)\omega^{-1} + f(\omega^2 z)\omega^{-2}\right] && (16)\\ &+\frac{1}{3}\left[f(z) + f(\omega z)\omega^{-2} + f(\omega^2 z)\omega^{-4}\right], && (17)\end{aligned}$$

which sums to $f(z)$ as required because, by Proposition 2 (see also Figure 1), the $f(\omega z)$ terms sum to 0, as do the $f(\omega^2 z)$ terms. The pattern holds for general n. A similar analysis that exploits the cyclic symmetry of ω can be used to show that each $f_j(z)$ is a harmonic of order j. □

This 'prototype' for decomposing functions into harmonics can be generalised in different ways, including the following discrete Fourier transform for a given integer n and $\omega_n = e^{2\pi i/n}$:

$$f(t) = \sum_{v=0}^{n-1} \hat{f}(v)\omega_n^{vt} \qquad (18)$$

$$\hat{f}(v) = \frac{1}{n}\sum_{t=0}^{n-1} f(t)\omega_n^{-vt}. \qquad (19)$$

To see its structure, consider the case of $n = 3$, where we have $\omega = e^{-2\pi i/3}$ and

$$\begin{aligned} f(t) = &\quad \hat{f}(0) + \hat{f}(1)\omega^t + \hat{f}(2)\omega^{2t} && (20)\\ = &\quad \frac{1}{3}\left[f(0)\omega^0 + f(1)\omega^0 + f(2)\omega^0\right] \\ &+\frac{1}{3}\left[f(0)\omega^t + f(1)\omega^{t-1} + f(2)\omega^{t-2}\right] \\ &+\frac{1}{3}\left[f(0)\omega^{2t} + f(1)\omega^{2t-2} + f(2)\omega^{2t-4}\right]. && (21)\end{aligned}$$

Observe that, for each t, if we substitute the actual t value on the RHS of (21), then only the $\frac{1}{3}f(t)$ terms remain and all the other $f(m), m \neq t$, terms sum to zero and disappear by Proposition 2. This patterns holds for general n and the use of (9) in the derivation of (10) can be seen as the limiting case of taking n to ∞ in the discrete Fourier transform.

Vector Space of Audio Signals

Another natural way to look at Fourier analysis is via linear algebra, where one can interpret (discrete) Fourier transform as a linear transformation between the standard basis (time and amplitude) to a second orthonormal basis (the sinusoids of the form $e^{-2\pi ik}$).

Let's start by considering continuous functions on $[0, 2\pi]$ denoted by $C[0, 2\pi]$. When we say any such function f can be rewritten as

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \sin(kx) + \sum_{m=1}^{\infty} b_m \cos(mx), \tag{22}$$

we basically mean that the set of functions

$$\{\lambda x.1, \lambda x. \sin(mx), \lambda x. \cos(mx) \mid m = 1, 2, 3, \ldots\} \tag{23}$$

forms an orthogonal basis for the space $C[0, 2\pi]$ endowed with the usual inner product and norm operators:

1. For all $f, g \in C[0, 2\pi]$, inner product is defined by $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)dx$.
2. For all $f \in C[0, 2\pi]$, the norm of f is defined by $||f||^2 = \langle f, f \rangle$.

To show that the basis given in (23) is orthogonal, we need to show that the inner product of any two basis elements is zero, which we can do using trigonometric identities like $\sin(x)\cos(y) = 0.5 \cdot [\sin(x + y) + \sin(x - y)]$ and $\cos(x)\cos(y) = 0.5 \cdot [\cos(x + y) + \cos(x - y)]$.

The coefficients in (22) can be obtained by projecting f onto the basis elements as usual, yielding

$$\begin{aligned}
a_0 &= \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx \\
a_k &= \frac{\langle f, \sin(kx) \rangle}{|| \sin(kx)||^2} = \frac{1}{\pi} \int_0^{2\pi} \sin(kx)f(x)dx \\
b_m &= \frac{\langle f, \cos(mx) \rangle}{|| \cos(mx)||^2} = \frac{1}{\pi} \int_0^{2\pi} \cos(mx)f(x)dx.
\end{aligned}$$

As noted before, (22) can be written in the form of $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ using Euler's formula, by equating $c_0 = \frac{1}{2}a_0, c_k = \frac{1}{2}(b_k - ia_k)$, and $c_{-k} = \frac{1}{2}(b_k + ia_k)$.

Finally, to extend the above to continuous functions on an arbitrary interval $[\alpha, \beta]$ that are discretely sampled at N equidistant points, we divide both $[\alpha, \beta]$ and $[0, 2\pi]$ into N equal subintervals and construct a one-on-one mapping between them.

Applications

The key application of Fourier transform in acoustics signal processing is that they simplify a lot of computations with audio signals, including convolutions of signals, and shifting and filtering of signals.

Limitations

In applications where we cannot sample enough data points from the signal of interest, performing Fourier transform by zero-ing out the unobserved coefficients can result in a lot of noise, and this phenomenon is commonly known as "leakage". Another limitation is that Fourier transforms yield mostly smooth functions so cannot handle functions with discontinuities well. Wavelet transforms are designed to deal with these limitations.

References

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